2-LOCAL DERIVATIONS ON VON NEUMANN ALGEBRAS OF TYPE I

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Abstract. In the present paper we prove that every 2-local derivation on a von Neumann algebra of type I is a derivation.

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Introduction

The present paper is devoted to 2-local derivations on von Neumann algebras. Recall that a 2-local derivation is defined as follows: given an algebra A, a map $\triangle : A \to A$ (not linear in general) is called a 2-local derivation if for every x, $y \in A$, there exists a derivation $D_{x,y} : A \to A$ such that $\triangle(x) = D_{x,y}(x)$ and $\triangle(y) = D_{x,y}(y)$.

In 1997, P. Šemrl [1] introduced the notion of 2-local derivations and described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H. A similar description for the finite-dimensional case appeared later in [2]. In the paper [3] 2-local derivations have been described on matrix algebras over finite-dimensional division rings.

In [4] the authors suggested a new technique and have generalized the above mentioned results of [1] and [2] for arbitrary Hilbert spaces. Namely they considered 2-local derivations on the algebra $B(H)$ of all linear bounded operators on an arbitrary (no separability is assumed) Hilbert space H and proved that every 2-local derivation on $B(H)$ is a derivation.

In the present paper we also suggest another technique and generalize the above mentioned results of [1], [2] and [4] for arbitrary von Neumann algebras of type I. Namely, we prove that every 2-local derivation on a von Neumann algebra of type I is a derivation.

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1. Preliminaries

Let M be a von Neumann algebra.

Definition. A linear map $D : M \to M$ is called a derivation, if $D(xy) = D(x)y + xD(y)$ for any two elements x, $y \in M$.

A map $\Delta : M \to M$ is called a 2-local derivation, if for any two elements x, $y \in M$ there exists a derivation $D_{x,y} : M \to M$ such that $\Delta(x) = D_{x,y}(x)$, $\Delta(y) = D_{x,y}(y)$.

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It is known that any derivation D on a von Neumann algebra M is an inner derivation, that is there exists an element $a \in M$ such that

$$D(x) = ax - xa, x \in M.$$

Therefore for a von Neumann algebra M the above definition is equivalent to the following one: A map $\Delta : M \to M$ is called a 2-local derivation, if for any two elements x, $y \in M$ there exists an element $a \in M$ such that $\Delta(x) = ax - xa$, $\Delta(y) = ay - ya$.

Further we will use the latter definition.

Let n be an arbitrary infinite cardinal number, Ξ be a set of indexes of the cardinality n. Let $\{e_{ij}\}$ be a set of matrix units such that e_{ij} is a $n \times n$-dimensional matrix, i.e. $e_{ij} = (a_{\alpha\beta})_{\alpha\beta\in\Xi}$, the (i,j)-th component of which is 1, i.e. $a_{ij} = 1$, and the rest components are zeros. Let $\{m_\xi\}_{\xi\in\Xi}$ be a set of $n \times n$-dimensional matrixes. By $\sum_{\xi\in\Xi} m_\xi$ we denote the matrix whose components are sums of the corresponding components of matrixes of the set $\{m_\xi\}_{\xi\in\Xi}$. Let

$$M_n(\mathbf{C}) = \{\{\lambda_{ij}e_{ij}\} : \mathit{for\ all\ indexes}\ i,\ j\ \lambda_{ij} \in \mathbf{C},$$

$$\mathit{and\ there\ exists\ such\ number}\ K \in \mathbf{R},\ \mathit{that\ for\ all}\ n \in N$$

$$\mathit{and}\ \{e_{kl}\}_{kl=1}^{n} \subseteq \{e_{ij}\} \| \sum_{kl=1}^{n} \lambda_{kl}e_{kl}\| \leq K\},$$

where $\| \ \|$ is a norm of a matrix. It is easy to see that $M_n(\mathbf{C})$ is a vector space.

The associative multiplication of elements in $M_n(\mathbf{C})$ can be defined as follows: if $x = \sum_{ij\in\Xi} \lambda_{ij}e_{ij}$, $y = \sum_{ij\in\Xi} \mu_{ij}e_{ij}$ are elements of $M_n(\mathbf{C})$ then $xy = \sum_{ij\in\Xi} \sum_{\xi\in\Xi} \lambda_{i\xi}\mu_{\xi j}e_{ij}$. With this operation $M_n(\mathbf{C})$ becomes an associative algebra and $M_n(\mathbf{C}) = B(l_2(\Xi))$, where $l_2(\Xi)$ is a Hilbert space over $\mathbf{C}$ with elements $\{x_i\}_{i\in\Xi}$, $x_i \in \mathbf{C}$ for all $i \in \Xi$, $B(l_2(\Xi))$ is the associative algebra of all bounded linear operators on the Hilbert space $l_2(\Xi)$. Then $M_n(\mathbf{C})$ is a von Neumann algebra of infinite $n \times n$-dimensional matrices over $\mathbf{C}$.

Similarly, if we take the algebra $B(H)$ of all bounded linear operators on an arbitrary Hilbert space H and if $\{q_i\}$ is an arbitrary maximal orthogonal set of minimal projections of the algebra $B(H)$, then $B(H) = \sum_{ij}^{\oplus} q_iB(H)q_j$ (see [5]).

Let X be a hyperstonean compact, and let $C(X)$ denote the commutative algebra of all complex-valued continuous functions on the compact X and

$$\mathcal{M} = \{\{\lambda_{ij}(x)e_{ij}\}_{ij\in\Xi} : (\forall ij\ \ \lambda_{ij}(x) \in C(X))$$

$$(\exists K \in R)(\forall m \in N)(\forall \{e_{kl}\}_{kl=1}^{m} \subseteq \{e_{ij}\})\| \sum_{kl=1\ldots m} \lambda_{kl}(x)e_{kl}\| \leq K\},$$

where $\| \sum_{kl=1\ldots m} \lambda_{kl}(x)e_{kl}\| \leq K$ means $(\forall x_o \in X)\| \sum_{kl=1\ldots m} \lambda_{kl}(x_o)e_{kl}\| \leq K$. The set $\mathcal{M}$ is a vector space with point-wise algebraic operations. The map $\| \ \| : \mathcal{M} \to \mathbf{R}_+$ defined as

$$\|a\| = \sup_{\{e_{kl}\}_{kl=1}^{n} \subseteq \{e_{ij}\}} \| \sum_{kl=1}^{n} \lambda_{kl}(x)e_{kl}\|,$$

is a norm on the vector space $\mathcal{M}$, where $a \in \mathcal{M}$ and $a = \sum_{ij\in\Xi} \lambda_{ij}(x)e_{ij}$.

Moreover $\mathcal{M}$ is a von Neumann algebra of type I_n and $\mathcal{M} = C(X) \otimes M_n(\mathbf{C})$, where the multiplication is defined as follows $xy = \sum_{ij\in\Xi} \sum_{\xi\in\Xi} \lambda_{i\xi}(x)\mu_{\xi j}(x)e_{ij}$ [6].

Let $\mathcal{M}$ be a von Neumann algebra, $\triangle : \mathcal{M} \to \mathcal{M}$ be a 2-local derivation. Now let us show that $\triangle$ is homogeneous. Indeed, for each $x \in \mathcal{M}$, and for $\lambda \in \mathbb{C}$ there exists a derivation $D_{x,\lambda x}$ such that $\triangle(x) = D_{x,\lambda x}(x)$ and $\triangle(\lambda x) = D_{x,\lambda x}(\lambda x)$. Then

$$\triangle(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \triangle (x).$$

Hence, $\triangle$ is homogenous. At the same time, for each $x \in \mathcal{M}$, there exists a derivation D_{x,x^2} such that $\triangle(x) = D_{x,x^2}(x)$ and $\triangle(x^2) = D_{x,x^2}(x^2)$. Then

$$\triangle(x^2) = D_{x,x^2}(x^2) = D_{x,x^2}(x)x + xD_{x,x^2}(x) = \triangle(x)x + x \triangle (x).$$

In [7] it is proved that any Jordan derivation on a semi-prime algebra is a derivation. Since $\mathcal{M}$ is semi-prime, the map $\triangle$ is a derivation if it is additive. Therefore, to prove that the 2-local derivation $\triangle : \mathcal{M} \to \mathcal{M}$ is a derivation it is sufficient to prove that $\triangle : \mathcal{M} \to \mathcal{M}$ is additive in the proofs of theorems 1 and 5.

2. 2-local derivations on von Neumann algebras of type I_n with an infinite cardinal number n

The following theorem is the key result of this section.

Theorem 1. *Let $\triangle : C(X) \otimes M_n(\mathbf{C}) \to C(X) \otimes M_n(\mathbf{C})$ be a 2-local derivation. Then $\triangle$ is a derivation.*

First let us prove lemmata which are necessary for the proof of theorem 1.

Put $\mathcal{M} = C(X) \otimes M_n(\mathbf{C})$, $e_{ij} := \mathbf{1}e_{ij}$ for all i, j, where $\mathbf{1}$ is unit of the algebra $C(X)$. Let $\{a(ij)\} \subset \mathcal{M}$ be the set such that

$$\triangle(e_{ij}) = a(ij)e_{ij} - e_{ij}a(ij).$$

for all i, j, put $a_{ij}e_{ij} = e_i a(ji)e_j$ for all pairs of different indexes i, j and let $\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}$ be the set of all such elements.

Lemma 2. *For any pair i, j of different indices the following equality holds*

$$\triangle(e_{ij}) = \{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta} + a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj}, \quad (1)$$

where $a(ij)_{ii}$, $a(ij)_{jj}$ are functions in $C(X)$ which are the coefficients of the Peirce components $e_{ii}a(ij)e_{ii}$, $e_{jj}a(ij)e_{jj}$.

Proof. Let k be an arbitrary index different from i, j and let $a(ij, ik) \in \mathcal{M}$ be an element such that

$$\triangle(e_{ik}) = a(ij, ik)e_{ik} - e_{ik}a(ij, ik) \text{ and } \triangle (e_{ij}) = a(ij, ik)e_{ij} - e_{ij}a(ij, ik).$$

Then

$$e_{kk} \triangle (e_{ij})e_{jj} = e_{kk}(a(ij, ik)e_{ij} - e_{ij}a(ij, ik))e_{jj} =$$
$$e_{kk}a(ij, ik)e_{ij} - 0 = e_{kk}a(ik)e_{ij} - e_{kk}e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{jj} =$$
$$e_{kk}a_{ki}e_{ij} - e_{kk}e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{jj} = e_{kk}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{kk}e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{jj} =$$
$$e_{kk}(\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta})e_{jj}.$$

Similarly,

$$e_{kk} \triangle (e_{ij})e_{ii} = e_{kk}(a(ij, ik)e_{ij} - e_{ij}a(ij, ik))e_{ii} =$$
$$e_{kk}a(ij, ik)e_{ij}e_{ii} - 0 = 0 - 0 = e_{kk}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij}e_{ii} - e_{kk}e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii} =$$
$$e_{kk}(\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta})e_{ii}.$$

Let $a(ij, kj) \in \mathcal{M}$ be an element such that

$$\triangle(e_{kj}) = a(ij, kj)e_{kj} - e_{kj}a(ij, kj) \text{ and } \triangle (e_{ij}) = a(ij, kj)e_{ij} - e_{ij}a(ij, kj).$$

Then

$$e_{ii} \triangle (e_{ij})e_{kk} = e_{ii}(a(ij,kj)e_{ij} - e_{ij}a(ij,kj))e_{kk} =$$
$$0 - e_{ij}a(ij,kj)e_{kk} = 0 - e_{ij}a(kj)e_{kk} = 0 - e_{ij}a_{jk}e_{kk} =$$
$$e_{ii}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij}e_{kk} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{kk} =$$
$$e_{ii}(\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta})e_{kk}.$$

Also we have

$$e_{jj} \triangle (e_{ij})e_{kk} = e_{jj}(a(ij,kj)e_{ij} - e_{ij}a(ij,kj))e_{kk} =$$
$$0 - 0 = e_{jj}\{a(ij)\}_{i\neq j}e_{ij}e_{kk} - e_{jj}e_{ij}\{a(ij)\}_{i\neq j}e_{kk} =$$
$$e_{jj}(\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta})e_{kk},$$

$$e_{ii} \triangle (e_{ij})e_{ii} = e_{ii}(a(ij)e_{ij} - e_{ij}a(ij))e_{ii} =$$
$$0 - e_{ij}a(ij)e_{ii} = 0 - e_{ij}a(ij)e_{ii} = 0 - e_{ij}a_{ji}e_{ii} =$$
$$e_{ii}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij}e_{ii} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii} =$$
$$e_{ii}(\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta})e_{ii}.$$

$$e_{jj} \triangle (e_{ij})e_{jj} = e_{jj}(a(ij)e_{ij} - e_{ij}a(ij))e_{jj} =$$
$$e_{jj}a(ij)e_{ij} - 0 = e_{jj}a_{ji}e_{ij} - 0 =$$
$$e_{jj}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{jj}e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{jj} =$$
$$e_{jj}(\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta})e_{jj}.$$

Hence the equality (1) holds. $\triangleright$

We take elements of the sets $\{\{e_{i\xi}\}_\xi\}_i$ and $\{\{e_{\xi j}\}_\xi\}_j$ in pairs $(\{e_{\alpha\xi}\}_\xi, \{e_{\xi\beta}\}_\xi)$ such that $\alpha \neq \beta$. Then using the set $\{(\{e_{\alpha\xi}\}_\xi, \{e_{\xi\beta}\}_\xi)\}$ of such pairs we get the set $\{e_{\alpha\beta}\}$.

Let $x = \{e_{\alpha\beta}\}$ be a set $\{v_{ij}e_{ij}\}_{ij}$ such that for all i, j if $(\alpha,\beta) \neq (i,j)$ then $v_{ij} = 0$ else $v_{ij} = 1$. Then $x \in \mathcal{M}$. Let $c \in \mathcal{M}$ be an element such that

$$\triangle(e_{ij}) = ce_{ij} - e_{ij}c \text{ and } \triangle(x) = cx - xc,$$

$i \neq j$ and fix the indices i, j.

Put $c = \{c_{ij}e_{ij}\} \in \mathcal{M}$ and $\bar{a} = \{a_{ij}e_{ij}\}_{i\neq j} \cup \{a_{ii}e_{ii}\}$, where $\{a_{ii}e_{ii}\} = \{c_{ii}e_{ii}\}$.

Lemma 3. *Let ξ, η be arbitrary different indices, and let $b \in \mathcal{M}$ be an element such that*

$$\triangle(e_{\xi\eta}) = be_{\xi\eta} - e_{\xi\eta}b \textit{ and } \triangle(x) = bx - xb.$$

Then $c_{\xi\xi} - c_{\eta\eta} = b_{\xi\xi} - b_{\eta\eta}$.

Proof. We have that there exist $\bar{\alpha}$, $\bar{\beta}$ such that $e_{\xi\bar{\alpha}}$, $e_{\bar{\beta}\eta} \in \{e_{\alpha\beta}\}$ (or $e_{\bar{\alpha}\eta}$, $e_{\xi\bar{\beta}} \in \{e_{\alpha\beta}\}$, or $e_{\bar{\alpha},\bar{\beta}} \in \{e_{\alpha\beta}\}$), and there exists a chain of pairs of indexes $(\hat{\alpha},\hat{\beta})$ in Ω, where $\Omega = \{(\check{\alpha},\check{\beta}) : e_{\check{\alpha},\check{\beta}} \in \{e_{\alpha\beta}\}\}$, connecting pairs $(\xi,\bar{\alpha})$, $(\bar{\beta},\eta)$ i.e.,

$$(\xi,\bar{\alpha}),(\bar{\alpha},\xi_1),(\xi_1,\eta_1),\ldots,(\eta_2,\bar{\beta}),(\bar{\beta},\eta).$$

Then

$$c_{\xi\xi} - c_{\bar{\alpha}\bar{\alpha}} = b_{\xi\xi} - b_{\bar{\alpha}\bar{\alpha}}, c_{\bar{\alpha}\bar{\alpha}} - c_{\xi_1\xi_1} = b_{\bar{\alpha}\bar{\alpha}} - b_{\xi_1\xi_1},$$
$$c_{\xi_1\xi_1} - c_{\eta_1\eta_1} = b_{\xi_1\xi_1} - b_{\eta_1\eta_1}, \ldots, c_{\eta_2\eta_2} - c_{\bar{\beta}\bar{\beta}} = b_{\eta_2\eta_2} - b_{\bar{\beta}\bar{\beta}}, c_{\bar{\beta}\bar{\beta}} - c_{\eta\eta} = b_{\bar{\beta}\bar{\beta}} - b_{\eta\eta}.$$

Hence

$$c_{\xi\xi} - b_{\xi\xi} = c_{\bar{\alpha}\bar{\alpha}} - b_{\bar{\alpha}\bar{\alpha}}, c_{\bar{\alpha}\bar{\alpha}} - b_{\bar{\alpha}\bar{\alpha}} = c_{\xi_1\xi_1} - b_{\xi_1\xi_1},$$
$$c_{\xi_1\xi_1} - b_{\xi_1\xi_1} = c_{\eta_1\eta_1} - b_{\eta_1\eta_1}, \ldots, c_{\eta_2\eta_2} - b_{\eta_2\eta_2} = c_{\bar{\beta}\bar{\beta}} - b_{\bar{\beta}\bar{\beta}}, c_{\bar{\beta}\bar{\beta}} - b_{\bar{\beta}\bar{\beta}} = c_{\eta\eta} - b_{\eta\eta}.$$

and $c_{\xi\xi} - b_{\xi\xi} = c_{\eta\eta} - b_{\eta\eta}$, $c_{\xi\xi} - c_{\eta\eta} = b_{\xi\xi} - b_{\eta\eta}$.

Therefore $c_{\xi\xi} - c_{\eta\eta} = b_{\xi\xi} - b_{\eta\eta}$. $\triangleright$

Lemma 4. *Let x be an element of the algebra $\mathcal{M}$. Then $\triangle(x) = \bar{a}x - x\bar{a}$, where $\bar{a}$ is defined as above.*

Proof. Let $d(ij) \in \mathcal{M}$ be an element such that

$$\triangle(e_{ij}) = d(ij)e_{ij} - e_{ij}d(ij) \text{ and } \triangle(x) = d(ij)x - xd(ij)$$

and $i \neq j$. Then

$$\triangle(e_{ij}) = d(ij)e_{ij} - e_{ij}d(ij) = e_{ii}d(ij)e_{ij} - e_{ij}d(ij)e_{jj} + (1-e_{ii})d(ij)e_{ij} - e_{ij}d(ij)(1-e_{jj}) =$$
$$a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj} + \{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}$$

for all i, j by lemma 2.

Since $e_{ii}d(ij)e_{ij} - e_{ij}d(ij)e_{jj} = a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj}$ we have

$$(1 - e_{ii})d(ij)e_{ii} = \{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii}, e_{jj}d(ij)(1 - e_{jj}) = e_{jj}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}$$

for all different i and j.

Hence by lemma 3 we have

$$e_{jj} \triangle (x)e_{ii} = e_{jj}(d(ij)x - xd(ij))e_{ii} =$$
$$e_{jj}d(ij)(1 - e_{jj})xe_{ii} + e_{jj}d(ij)e_{jj}xe_{ii} - e_{jj}x(1 - e_{ii})d(ij)e_{ii} - e_{jj}xe_{ii}d(ij)e_{ii} =$$
$$e_{jj}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}xe_{ii} - e_{jj}x\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii} + e_{jj}d(ij)e_{jj}xe_{ii} - e_{jj}xe_{ii}d(ij)e_{ii} =$$
$$e_{jj}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}xe_{ii} - e_{jj}x\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii} + c_{jj}e_{jj}xe_{ii} - e_{jj}xe_{ii}c_{ii}e_{ii}$$

since $d(ij)_{jj} - d(ij)_{ii} = b_{jj} - b_{ii}$.

Hence

$$e_{ii} \triangle (x)e_{ii} = e_{ii}(d(ij)x - xd(ij))e_{ii} =$$
$$e_{ii}d(ij)(1 - e_{ii})xe_{ii} + e_{ii}d(ij)e_{ii}xe_{ii} - e_{ii}x(1 - e_{ii})d(ij)e_{ii} - e_{ii}xe_{ii}d(ij)e_{ii} =$$
$$e_{ii}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}xe_{ii} - e_{ii}x\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii} + e_{ii}d(ij)e_{ii}xe_{ii} - e_{ii}xe_{ii}d(ij)e_{ii} =$$
$$e_{ii}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}xe_{ii} - e_{ii}x\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii} + 0 = c_{ii}e_{ii}xe_{ii} - e_{ii}xc_{ii}e_{ii}.$$

Hence

$$\triangle(x) = (\sum_i c_{ii}e_{ii})x - x(\sum_i c_{ii}e_{ii}) + \{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}x - x\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta} =$$
$$\bar{a}x - x\bar{a}$$

for all $x \in \mathcal{M}$. $\triangleright$

Proof of theorem 1. Let $V = \{\{\lambda_{ij}e_{ij}\}_{ij} : \{\lambda_{ij}\} \subset C(X)\}$ (the set of all infinite $n \times n$-dimensional function-valued matrices). Then V is a vector space with componentwise algebraic operations and $\mathcal{M}$ is a vector subspace of V.

By lemma 4 $\triangle(e_{ii}) = \bar{a}e_{ii} - e_{ii}\bar{a} \in \mathcal{M}$. Hence

$$\sum_\xi a_{\xi i}e_{\xi i} - \sum_\xi a_{i\xi}e_{i\xi} \in \mathcal{M}.$$

Then

$$e_{ii}(\sum_\xi a_{\xi i}e_{\xi i} - \sum_\xi a_{i\xi}e_{i\xi}) = a_{ii}e_{ii} - \sum_\xi a_{i\xi}e_{i\xi} \in \mathcal{M}$$

and

$$(\sum_\xi a_{\xi i}e_{\xi i} - \sum_\xi a_{i\xi}e_{i\xi})e_{ii} = \sum_\xi a_{\xi i}e_{\xi i} - a_{ii}e_{ii} \in \mathcal{M}.$$

Therefore $\sum_\xi a_{\xi i}e_{\xi i}$, $\sum_\xi a_{i\xi}e_{i\xi} \in \mathcal{M}$ i.e., $\bar{a}e_{ii}, e_{ii}\bar{a} \in \mathcal{M}$. Hence $e_{ii}\bar{a}x, x\bar{a}e_{ii} \in \mathcal{M}$ for any i and

$$\bar{a}x, x\bar{a} \in V$$

for any element $x = \{x_{ij}e_{ij}\} \in \mathcal{M}$, i.e.,

$$\sum_\xi a_{i\xi}x_{\xi j}e_{ij}, \sum_\xi x_{i\xi}a_{\xi j}e_{ij} \in \mathbb{C}e_{ij}$$

for all i, j. Therefore for all x, $y \in \mathcal{M}$ we have that the elements $\bar{a}x$, $x\bar{a}$, $\bar{a}y$, $y\bar{a}$, $\bar{a}(x+y)$, $(x+y)\bar{a}$ belong to V. Hence

$$\triangle(x+y) = \triangle(x) + \triangle(y)$$

by lemma 4.

Similarly for all x, $y \in \mathcal{M}$ we have

$$(\bar{a}x + x\bar{a})y = \bar{a}xy - x\bar{a}y \in \mathcal{M}, \bar{a}xy = \bar{a}(xy) \in V.$$

Then $x\bar{a}y = \bar{a}xy - (\bar{a}x - x\bar{a})y$ and $x\bar{a}y \in V$. Therefore

$$\bar{a}(xy) - (xy)\bar{a} = \bar{a}xy - x\bar{a}y + x\bar{a}y - xy\bar{a} = (\bar{a}x - x\bar{a})y + x(\bar{a}y - y\bar{a}).$$

Hence

$$\triangle(xy) = \triangle(x)y + x \triangle (y)$$

by lemma 4. Now we show that $\triangle$ is homogeneous. Indeed, for each $x \in \mathcal{M}$, and for $\lambda \in \mathbb{C}$ there exists a derivation $D_{x,\lambda x}$ such that $\triangle(x) = D_{x,\lambda x}(x)$ and $\triangle(\lambda x) = D_{x,\lambda x}(\lambda x)$. Then

$$\triangle(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \triangle (x).$$

Hence, $\triangle$ is homogenous and therefore it is a linear operator and a derivation. The proof is complete.

$\triangleright$

3. The main theorem

Theorem 5. *Let M be a von Neumann algebra of type I and let $\triangle : M \to M$ be a 2-local derivation. Then $\triangle$ is a derivation.*

Proof. We have that

$$M = \sum_j^{\oplus} M_{I_{n_j}},$$

where $M_{I_{n_j}}$ is a von Neumann algebra of type I_{n_j}, n_j is a cardinal number for any j. Let $x_j \in M_{I_{n_j}}$ for any j and $x = \sum_j x_j$. Note that $\triangle(x_j) \in M_{I_{n_j}}$ for all $x_j \in M_{I_{n_j}}$. Hence

$$\triangle|_{M_{I_{n_j}}} : M_{I_{n_j}} \to M_{I_{n_j}}$$

and $\triangle$ is a 2-local derivation on $M_{I_{n_j}}$. There exists a hyperstonean compact X such that $M_{I_{n_j}} \cong C(X) \otimes M_{n_j}(\mathbf{C})$. Hence by theorem 1 $\triangle$ is a derivation on $M_{I_{n_j}}$.

Let x be an arbitrary element of M. Then there exists $d(j) \in M$ such that $\triangle(x) = d(j)x - xd(j)$, $\triangle(x_j) = d(j)x_j - x_jd(j)$ and

$$z_j \triangle (x) = z_j(d(j)x - xd(j)) = z_j \sum_i (d(j)x_i - x_id(j)) =$$

$$d(j)x_j - x_jd(j) = \triangle(x_j),$$

for all j, where z_j is unit of $M_{I_{n_j}}$. Hence

$$\triangle(x) = \sum_j z_j \triangle (x) = \sum_j \triangle(x_j).$$

Since x was chosen arbitrarily $\triangle$ is a derivation on M by the last equality.

Indeed, let $x, y \in M$. Then

$$\triangle(x) + \triangle(y) = \sum_j \triangle(x_j) + \sum_j \triangle(y_j) = \sum_j [\triangle(x_j) + \triangle(y_j)] =$$

$$\sum_j \triangle(x_j + y_j) = \sum_j z_j \triangle (x + y) = \triangle(x + y).$$

Similarly,

$$\triangle(xy) = \sum_j \triangle(x_j y_j) = \sum_j [\triangle(x_j) y_j + x_j \triangle (y_j)] =$$

$$\sum_j \triangle(x_j) y_j + \sum_j x_j \triangle (y_j) = \sum_j \triangle(x_j) \sum_j y_j + \sum_j x_j \sum_j \triangle(y_j) =$$

$$\triangle(x) y + x \triangle (y).$$

By the proof of the previous theorem $\triangle$ is homogenous. Hence $\triangle$ is a linear operator and a derivation. The proof is complete. $\triangleright$

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2-LOCAL DERIVATIONS ON VON NEUMANN ALGEBRAS OF TYPE I

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Abstract. In the present paper we prove that every 2-local derivation on a von Neumann algebra of type I is a derivation.

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Introduction

The present paper is devoted to 2-local derivations on von Neumann algebras. Recall that a 2-local derivation is defined as follows: given an algebra A, a map $\triangle : A \to A$ (not linear in general) is called a 2-local derivation if for every x, $y \in A$, there exists a derivation $D_{x,y} : A \to A$ such that $\triangle(x) = D_{x,y}(x)$ and $\triangle(y) = D_{x,y}(y)$.

In 1997, P. Šemrl [1] introduced the notion of 2-local derivations and described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H. A similar description for the finite-dimensional case appeared later in [2]. In the paper [3] 2-local derivations have been described on matrix algebras over finite-dimensional division rings.

In [4] the authors suggested a new technique and have generalized the above mentioned results of [1] and [2] for arbitrary Hilbert spaces. Namely they considered 2-local derivations on the algebra $B(H)$ of all linear bounded operators on an arbitrary (no separability is assumed) Hilbert space H and proved that every 2-local derivation on $B(H)$ is a derivation.

In the present paper we also suggest another technique and generalize the above mentioned results of [1], [2] and [4] for arbitrary von Neumann algebras of type I. Namely, we prove that every 2-local derivation on a von Neumann algebra of type I is a derivation.

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1. Preliminaries

Let M be a von Neumann algebra.

Definition. A linear map $D : M \to M$ is called a derivation, if $D(xy) = D(x)y + xD(y)$ for any two elements x, $y \in M$.

A map $\Delta : M \to M$ is called a 2-local derivation, if for any two elements x, $y \in M$ there exists a derivation $D_{x,y} : M \to M$ such that $\Delta(x) = D_{x,y}(x)$, $\Delta(y) = D_{x,y}(y)$.

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It is known that any derivation D on a von Neumann algebra M is an inner derivation, that is there exists an element $a \in M$ such that

$$D(x) = ax - xa, x \in M.$$

Therefore for a von Neumann algebra M the above definition is equivalent to the following one: A map $\Delta : M \to M$ is called a 2-local derivation, if for any two elements $x,\ y \in M$ there exists an element $a \in M$ such that $\Delta(x) = ax - xa$, $\Delta(y) = ay - ya$.

Further we will use the latter definition.

Let n be an arbitrary infinite cardinal number, Ξ be a set of indexes of the cardinality n. Let $\{e_{ij}\}$ be a set of matrix units such that e_{ij} is a $n \times n$-dimensional matrix, i.e. $e_{ij} = (a_{\alpha\beta})_{\alpha\beta\in\Xi}$, the (i, j)-th component of which is 1, i.e. $a_{ij} = 1$, and the rest components are zeros. Let $\{m_\xi\}_{\xi\in\Xi}$ be a set of $n \times n$-dimensional matrixes. By $\sum_{\xi\in\Xi} m_\xi$ we denote the matrix whose components are sums of the corresponding components of matrixes of the set $\{m_\xi\}_{\xi\in\Xi}$. Let

$$M_n(\mathbf{C}) = \{\{\lambda_{ij}e_{ij}\} : \mathit{for\ all\ indexes}\ i,\ j\ \lambda_{ij} \in \mathbf{C},$$

$$\mathit{and\ there\ exists\ such\ number}\ K \in \mathbf{R},\ \mathit{that\ for\ all}\ n \in N$$

$$\mathit{and}\ \{e_{kl}\}_{kl=1}^{n} \subseteq \{e_{ij}\} \| \sum_{kl=1}^{n} \lambda_{kl}e_{kl}\| \le K\},$$

where $\|\ \|$ is a norm of a matrix. It is easy to see that $M_n(\mathbf{C})$ is a vector space.

The associative multiplication of elements in $M_n(\mathbf{C})$ can be defined as follows: if $x = \sum_{ij\in\Xi} \lambda_{ij}e_{ij}$, $y = \sum_{ij\in\Xi} \mu_{ij}e_{ij}$ are elements of $M_n(\mathbf{C})$ then $xy = \sum_{ij\in\Xi}\sum_{\xi\in\Xi} \lambda_{i\xi}\mu_{\xi j}e_{ij}$. With this operation $M_n(\mathbf{C})$ becomes an associative algebra and $M_n(\mathbf{C}) = B(l_2(\Xi))$, where $l_2(\Xi)$ is a Hilbert space over $\mathbf{C}$ with elements $\{x_i\}_{i\in\Xi}$, $x_i \in \mathbf{C}$ for all $i \in \Xi$, $B(l_2(\Xi))$ is the associative algebra of all bounded linear operators on the Hilbert space $l_2(\Xi)$. Then $M_n(\mathbf{C})$ is a von Neumann algebra of infinite $n \times n$-dimensional matrices over $\mathbf{C}$.

Similarly, if we take the algebra $B(H)$ of all bounded linear operators on an arbitrary Hilbert space H and if $\{q_i\}$ is an arbitrary maximal orthogonal set of minimal projections of the algebra $B(H)$, then $B(H) = \sum_{ij}^{\oplus} q_iB(H)q_j$ (see [5]).

Let X be a hyperstonean compact, and let $C(X)$ denote the commutative algebra of all complex-valued continuous functions on the compact X and

$$\mathcal{M} = \{\{\lambda_{ij}(x)e_{ij}\}_{ij\in\Xi} : (\forall ij\ \ \lambda_{ij}(x) \in C(X))$$

$$(\exists K \in R)(\forall m \in N)(\forall\{e_{kl}\}_{kl=1}^{m} \subseteq \{e_{ij}\})\| \sum_{kl=1\dots m} \lambda_{kl}(x)e_{kl}\| \le K\},$$

where $\|\sum_{kl=1\dots m} \lambda_{kl}(x)e_{kl}\| \le K$ means $(\forall x_o \in X)\|\sum_{kl=1\dots m} \lambda_{kl}(x_o)e_{kl}\| \le K$. The set $\mathcal{M}$ is a vector space with point-wise algebraic operations. The map $\|\ \| : \mathcal{M} \to \mathbf{R}_+$ defined as

$$\|a\| = \sup_{\{e_{kl}\}_{kl=1}^{n}\subseteq\{e_{ij}\}} \|\sum_{kl=1}^{n} \lambda_{kl}(x)e_{kl}\|,$$

is a norm on the vector space $\mathcal{M}$, where $a \in \mathcal{M}$ and $a = \sum_{ij\in\Xi} \lambda_{ij}(x)e_{ij}$.

Moreover $\mathcal{M}$ is a von Neumann algebra of type I_n and $\mathcal{M} = C(X) \otimes M_n(\mathbf{C})$, where the multiplication is defined as follows $xy = \sum_{ij\in\Xi}\sum_{\xi\in\Xi} \lambda_{i\xi}(x)\mu_{\xi j}(x)e_{ij}$ [6].

Let $\mathcal{M}$ be a von Neumann algebra, $\triangle : \mathcal{M} \to \mathcal{M}$ be a 2-local derivation. Now let us show that $\triangle$ is homogeneous. Indeed, for each $x \in \mathcal{M}$, and for $\lambda \in \mathbb{C}$ there exists a derivation $D_{x,\lambda x}$ such that $\triangle(x) = D_{x,\lambda x}(x)$ and $\triangle(\lambda x) = D_{x,\lambda x}(\lambda x)$. Then

$$\triangle(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \triangle (x).$$

Hence, $\triangle$ is homogenous. At the same time, for each $x \in \mathcal{M}$, there exists a derivation D_{x,x^2} such that $\triangle(x) = D_{x,x^2}(x)$ and $\triangle(x^2) = D_{x,x^2}(x^2)$. Then

$$\triangle(x^2) = D_{x,x^2}(x^2) = D_{x,x^2}(x)x + xD_{x,x^2}(x) = \triangle(x)x + x \triangle (x).$$

In [7] it is proved that any Jordan derivation on a semi-prime algebra is a derivation. Since $\mathcal{M}$ is semi-prime, the map $\triangle$ is a derivation if it is additive. Therefore, to prove that the 2-local derivation $\triangle : \mathcal{M} \to \mathcal{M}$ is a derivation it is sufficient to prove that $\triangle : \mathcal{M} \to \mathcal{M}$ is additive in the proofs of theorems 1 and 5.

2. 2-local derivations on von Neumann algebras of type I_n with an infinite cardinal number n

The following theorem is the key result of this section.

Theorem 1. *Let $\triangle : C(X) \otimes M_n(\mathbf{C}) \to C(X) \otimes M_n(\mathbf{C})$ be a 2-local derivation. Then $\triangle$ is a derivation.*

First let us prove lemmata which are necessary for the proof of theorem 1.

Put $\mathcal{M} = C(X) \otimes M_n(\mathbf{C})$, $e_{ij} := \mathbf{1}e_{ij}$ for all i, j, where $\mathbf{1}$ is unit of the algebra $C(X)$. Let $\{a(ij)\} \subset \mathcal{M}$ be the set such that

$$\triangle(e_{ij}) = a(ij)e_{ij} - e_{ij}a(ij).$$

for all i, j, put $a_{ij}e_{ij} = e_i a(ji)e_j$ for all pairs of different indexes i, j and let $\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}$ be the set of all such elements.

Lemma 2. *For any pair i, j of different indices the following equality holds*

$$\triangle(e_{ij}) = \{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta} + a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj}, \quad (1)$$

where $a(ij)_{ii}$, $a(ij)_{jj}$ are functions in $C(X)$ which are the coefficients of the Peirce components $e_{ii}a(ij)e_{ii}$, $e_{jj}a(ij)e_{jj}$.

Proof. Let k be an arbitrary index different from i, j and let $a(ij, ik) \in \mathcal{M}$ be an element such that

$$\triangle(e_{ik}) = a(ij, ik)e_{ik} - e_{ik}a(ij, ik) \text{ and } \triangle (e_{ij}) = a(ij, ik)e_{ij} - e_{ij}a(ij, ik).$$

Then

$$e_{kk} \triangle (e_{ij})e_{jj} = e_{kk}(a(ij, ik)e_{ij} - e_{ij}a(ij, ik))e_{jj} =$$
$$e_{kk}a(ij, ik)e_{ij} - 0 = e_{kk}a(ik)e_{ij} - e_{kk}e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{jj} =$$
$$e_{kk}a_{ki}e_{ij} - e_{kk}e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{jj} = e_{kk}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{kk}e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{jj} =$$
$$e_{kk}(\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta})e_{jj}.$$

Similarly,

$$e_{kk} \triangle (e_{ij})e_{ii} = e_{kk}(a(ij, ik)e_{ij} - e_{ij}a(ij, ik))e_{ii} =$$
$$e_{kk}a(ij, ik)e_{ij}e_{ii} - 0 = 0 - 0 = e_{kk}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij}e_{ii} - e_{kk}e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii} =$$
$$e_{kk}(\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta})e_{ii}.$$

Let $a(ij, kj) \in \mathcal{M}$ be an element such that

$$\triangle(e_{kj}) = a(ij, kj)e_{kj} - e_{kj}a(ij, kj) \text{ and } \triangle (e_{ij}) = a(ij, kj)e_{ij} - e_{ij}a(ij, kj).$$

Then

$$e_{ii} \triangle (e_{ij})e_{kk} = e_{ii}(a(ij,kj)e_{ij} - e_{ij}a(ij,kj))e_{kk} =$$
$$0 - e_{ij}a(ij,kj)e_{kk} = 0 - e_{ij}a(kj)e_{kk} = 0 - e_{ij}a_{jk}e_{kk} =$$
$$e_{ii}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij}e_{kk} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{kk} =$$
$$e_{ii}(\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta})e_{kk}.$$

Also we have

$$e_{jj} \triangle (e_{ij})e_{kk} = e_{jj}(a(ij,kj)e_{ij} - e_{ij}a(ij,kj))e_{kk} =$$
$$0 - 0 = e_{jj}\{a(ij)\}_{i\neq j}e_{ij}e_{kk} - e_{jj}e_{ij}\{a(ij)\}_{i\neq j}e_{kk} =$$
$$e_{jj}(\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta})e_{kk},$$

$$e_{ii} \triangle (e_{ij})e_{ii} = e_{ii}(a(ij)e_{ij} - e_{ij}a(ij))e_{ii} =$$
$$0 - e_{ij}a(ij)e_{ii} = 0 - e_{ij}a(ij)e_{ii} = 0 - e_{ij}a_{ji}e_{ii} =$$
$$e_{ii}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij}e_{ii} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii} =$$
$$e_{ii}(\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta})e_{ii}.$$

$$e_{jj} \triangle (e_{ij})e_{jj} = e_{jj}(a(ij)e_{ij} - e_{ij}a(ij))e_{jj} =$$
$$e_{jj}a(ij)e_{ij} - 0 = e_{jj}a_{ji}e_{ij} - 0 =$$
$$e_{jj}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{jj}e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{jj} =$$
$$e_{jj}(\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta})e_{jj}.$$

Hence the equality (1) holds. ▹

We take elements of the sets $\{\{e_{i\xi}\}_\xi\}_i$ and $\{\{e_{\xi j}\}_\xi\}_j$ in pairs $(\{e_{\alpha\xi}\}_\xi, \{e_{\xi\beta}\}_\xi)$ such that $\alpha \neq \beta$. Then using the set $\{(\{e_{\alpha\xi}\}_\xi, \{e_{\xi\beta}\}_\xi)\}$ of such pairs we get the set $\{e_{\alpha\beta}\}$.

Let $x = \{e_{\alpha\beta}\}$ be a set $\{v_{ij}e_{ij}\}_{ij}$ such that for all i, j if $(\alpha,\beta) \neq (i,j)$ then $v_{ij} = 0$ else $v_{ij} = 1$. Then $x \in \mathcal{M}$. Let $c \in \mathcal{M}$ be an element such that

$$\triangle(e_{ij}) = ce_{ij} - e_{ij}c \text{ and } \triangle(x) = cx - xc,$$

$i \neq j$ and fix the indices i, j.

Put $c = \{c_{ij}e_{ij}\} \in \mathcal{M}$ and $\bar{a} = \{a_{ij}e_{ij}\}_{i\neq j} \cup \{a_{ii}e_{ii}\}$, where $\{a_{ii}e_{ii}\} = \{c_{ii}e_{ii}\}$.

Lemma 3. *Let ξ, η be arbitrary different indices, and let $b \in \mathcal{M}$ be an element such that*

$$\triangle(e_{\xi\eta}) = be_{\xi\eta} - e_{\xi\eta}b \textit{ and } \triangle(x) = bx - xb.$$

Then $c_{\xi\xi} - c_{\eta\eta} = b_{\xi\xi} - b_{\eta\eta}$.

Proof. We have that there exist $\bar{\alpha}$, $\bar{\beta}$ such that $e_{\xi\bar{\alpha}}$, $e_{\bar{\beta}\eta} \in \{e_{\alpha\beta}\}$ (or $e_{\bar{\alpha}\eta}$, $e_{\xi\bar{\beta}} \in \{e_{\alpha\beta}\}$, or $e_{\bar{\alpha},\bar{\beta}} \in \{e_{\alpha\beta}\}$), and there exists a chain of pairs of indexes $(\hat{\alpha},\hat{\beta})$ in Ω, where $\Omega = \{(\check{\alpha},\check{\beta}) : e_{\check{\alpha},\check{\beta}} \in \{e_{\alpha\beta}\}\}$, connecting pairs $(\xi,\bar{\alpha})$, $(\bar{\beta},\eta)$ i.e.,

$$(\xi,\bar{\alpha}), (\bar{\alpha},\xi_1), (\xi_1,\eta_1), \ldots, (\eta_2,\bar{\beta}), (\bar{\beta},\eta).$$

Then

$$c_{\xi\xi} - c_{\bar{\alpha}\bar{\alpha}} = b_{\xi\xi} - b_{\bar{\alpha}\bar{\alpha}}, c_{\bar{\alpha}\bar{\alpha}} - c_{\xi_1\xi_1} = b_{\bar{\alpha}\bar{\alpha}} - b_{\xi_1\xi_1},$$
$$c_{\xi_1\xi_1} - c_{\eta_1\eta_1} = b_{\xi_1\xi_1} - b_{\eta_1\eta_1}, \ldots, c_{\eta_2\eta_2} - c_{\bar{\beta}\bar{\beta}} = b_{\eta_2\eta_2} - b_{\bar{\beta}\bar{\beta}}, c_{\bar{\beta}\bar{\beta}} - c_{\eta\eta} = b_{\bar{\beta}\bar{\beta}} - b_{\eta\eta}.$$

Hence

$$c_{\xi\xi} - b_{\xi\xi} = c_{\bar{\alpha}\bar{\alpha}} - b_{\bar{\alpha}\bar{\alpha}}, c_{\bar{\alpha}\bar{\alpha}} - b_{\bar{\alpha}\bar{\alpha}} = c_{\xi_1\xi_1} - b_{\xi_1\xi_1},$$
$$c_{\xi_1\xi_1} - b_{\xi_1\xi_1} = c_{\eta_1\eta_1} - b_{\eta_1\eta_1}, \ldots, c_{\eta_2\eta_2} - b_{\eta_2\eta_2} = c_{\bar{\beta}\bar{\beta}} - b_{\bar{\beta}\bar{\beta}}, c_{\bar{\beta}\bar{\beta}} - b_{\bar{\beta}\bar{\beta}} = c_{\eta\eta} - b_{\eta\eta}.$$

and $c_{\xi\xi} - b_{\xi\xi} = c_{\eta\eta} - b_{\eta\eta}$, $c_{\xi\xi} - c_{\eta\eta} = b_{\xi\xi} - b_{\eta\eta}$.

Therefore $c_{\xi\xi} - c_{\eta\eta} = b_{\xi\xi} - b_{\eta\eta}$. $\triangleright$

Lemma 4. *Let x be an element of the algebra $\mathcal{M}$. Then $\triangle(x) = \bar{a}x - x\bar{a}$, where $\bar{a}$ is defined as above.*

Proof. Let $d(ij) \in \mathcal{M}$ be an element such that

$$\triangle(e_{ij}) = d(ij)e_{ij} - e_{ij}d(ij) \text{ and } \triangle(x) = d(ij)x - xd(ij)$$

and $i \neq j$. Then

$$\triangle(e_{ij}) = d(ij)e_{ij} - e_{ij}d(ij) = e_{ii}d(ij)e_{ij} - e_{ij}d(ij)e_{jj} + (1-e_{ii})d(ij)e_{ij} - e_{ij}d(ij)(1-e_{jj}) =$$
$$a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj} + \{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ij} - e_{ij}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}$$

for all i, j by lemma 2.

Since $e_{ii}d(ij)e_{ij} - e_{ij}d(ij)e_{jj} = a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj}$ we have

$$(1-e_{ii})d(ij)e_{ii} = \{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii}, e_{jj}d(ij)(1-e_{jj}) = e_{jj}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}$$

for all different i and j.

Hence by lemma 3 we have

$$e_{jj} \triangle (x)e_{ii} = e_{jj}(d(ij)x - xd(ij))e_{ii} =$$
$$e_{jj}d(ij)(1-e_{jj})xe_{ii} + e_{jj}d(ij)e_{jj}xe_{ii} - e_{jj}x(1-e_{ii})d(ij)e_{ii} - e_{jj}xe_{ii}d(ij)e_{ii} =$$
$$e_{jj}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}xe_{ii} - e_{jj}x\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii} + e_{jj}d(ij)e_{jj}xe_{ii} - e_{jj}xe_{ii}d(ij)e_{ii} =$$
$$e_{jj}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}xe_{ii} - e_{jj}x\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii} + c_{jj}e_{jj}xe_{ii} - e_{jj}xe_{ii}c_{ii}e_{ii}$$

since $d(ij)_{jj} - d(ij)_{ii} = b_{jj} - b_{ii}$.

Hence

$$e_{ii} \triangle (x)e_{ii} = e_{ii}(d(ij)x - xd(ij))e_{ii} =$$
$$e_{ii}d(ij)(1-e_{ii})xe_{ii} + e_{ii}d(ij)e_{ii}xe_{ii} - e_{ii}x(1-e_{ii})d(ij)e_{ii} - e_{ii}xe_{ii}d(ij)e_{ii} =$$
$$e_{ii}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}xe_{ii} - e_{ii}x\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii} + e_{ii}d(ij)e_{ii}xe_{ii} - e_{ii}xe_{ii}d(ij)e_{ii} =$$
$$e_{ii}\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}xe_{ii} - e_{ii}x\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}e_{ii} + 0 = c_{ii}e_{ii}xe_{ii} - e_{ii}xc_{ii}e_{ii}.$$

Hence

$$\triangle(x) = (\sum_i c_{ii}e_{ii})x - x(\sum_i c_{ii}e_{ii}) + \{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta}x - x\{a_{\xi\eta}e_{\xi\eta}\}_{\xi\neq\eta} =$$
$$\bar{a}x - x\bar{a}$$

for all $x \in \mathcal{M}$. $\triangleright$

Proof of theorem 1. Let $V = \{\{\lambda_{ij}e_{ij}\}_{ij} : \{\lambda_{ij}\} \subset C(X)\}$ (the set of all infinite $n \times n$-dimensional function-valued matrices). Then V is a vector space with componentwise algebraic operations and $\mathcal{M}$ is a vector subspace of V.

By lemma 4 $\triangle(e_{ii}) = \bar{a}e_{ii} - e_{ii}\bar{a} \in \mathcal{M}$. Hence

$$\sum_\xi a_{\xi i}e_{\xi i} - \sum_\xi a_{i\xi}e_{i\xi} \in \mathcal{M}.$$

Then

$$e_{ii}(\sum_\xi a_{\xi i}e_{\xi i} - \sum_\xi a_{i\xi}e_{i\xi}) = a_{ii}e_{ii} - \sum_\xi a_{i\xi}e_{i\xi} \in \mathcal{M}$$

and

$$(\sum_\xi a_{\xi i}e_{\xi i} - \sum_\xi a_{i\xi}e_{i\xi})e_{ii} = \sum_\xi a_{\xi i}e_{\xi i} - a_{ii}e_{ii} \in \mathcal{M}.$$

Therefore $\sum_\xi a_{\xi i}e_{\xi i}$, $\sum_\xi a_{i\xi}e_{i\xi} \in \mathcal{M}$ i.e., $\bar{a}e_{ii}, e_{ii}\bar{a} \in \mathcal{M}$. Hence $e_{ii}\bar{a}x, x\bar{a}e_{ii} \in \mathcal{M}$ for any i and

$$\bar{a}x, x\bar{a} \in V$$

for any element $x = \{x_{ij}e_{ij}\} \in \mathcal{M}$, i.e.,

$$\sum_\xi a_{i\xi}x_{\xi j}e_{ij}, \sum_\xi x_{i\xi}a_{\xi j}e_{ij} \in \mathbb{C}e_{ij}$$

for all i, j. Therefore for all x, $y \in \mathcal{M}$ we have that the elements $\bar{a}x$, $x\bar{a}$, $\bar{a}y$, $y\bar{a}$, $\bar{a}(x+y)$, $(x+y)\bar{a}$ belong to V. Hence

$$\triangle(x+y) = \triangle(x) + \triangle(y)$$

by lemma 4.

Similarly for all x, $y \in \mathcal{M}$ we have

$$(\bar{a}x + x\bar{a})y = \bar{a}xy - x\bar{a}y \in \mathcal{M}, \bar{a}xy = \bar{a}(xy) \in V.$$

Then $x\bar{a}y = \bar{a}xy - (\bar{a}x - x\bar{a})y$ and $x\bar{a}y \in V$. Therefore

$$\bar{a}(xy) - (xy)\bar{a} = \bar{a}xy - x\bar{a}y + x\bar{a}y - xy\bar{a} = (\bar{a}x - x\bar{a})y + x(\bar{a}y - y\bar{a}).$$

Hence

$$\triangle(xy) = \triangle(x)y + x \triangle (y)$$

by lemma 4. Now we show that $\triangle$ is homogeneous. Indeed, for each $x \in \mathcal{M}$, and for $\lambda \in \mathbb{C}$ there exists a derivation $D_{x,\lambda x}$ such that $\triangle(x) = D_{x,\lambda x}(x)$ and $\triangle(\lambda x) = D_{x,\lambda x}(\lambda x)$. Then

$$\triangle(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \triangle (x).$$

Hence, $\triangle$ is homogenous and therefore it is a linear operator and a derivation. The proof is complete.

$\triangleright$

3. The main theorem

Theorem 5. *Let M be a von Neumann algebra of type I and let $\triangle : M \to M$ be a 2-local derivation. Then $\triangle$ is a derivation.*

Proof. We have that

$$M = \sum_j^{\oplus} M_{I_{n_j}},$$

where $M_{I_{n_j}}$ is a von Neumann algebra of type I_{n_j}, n_j is a cardinal number for any j. Let $x_j \in M_{I_{n_j}}$ for any j and $x = \sum_j x_j$. Note that $\triangle(x_j) \in M_{I_{n_j}}$ for all $x_j \in M_{I_{n_j}}$. Hence

$$\triangle|_{M_{I_{n_j}}} : M_{I_{n_j}} \to M_{I_{n_j}}$$

and $\triangle$ is a 2-local derivation on $M_{I_{n_j}}$. There exists a hyperstonean compact X such that $M_{I_{n_j}} \cong C(X) \otimes M_{n_j}(\mathbf{C})$. Hence by theorem 1 $\triangle$ is a derivation on $M_{I_{n_j}}$.

Let x be an arbitrary element of M. Then there exists $d(j) \in M$ such that $\triangle(x) = d(j)x - xd(j)$, $\triangle(x_j) = d(j)x_j - x_jd(j)$ and

$$z_j \triangle (x) = z_j(d(j)x - xd(j)) = z_j \sum_i (d(j)x_i - x_id(j)) =$$

$$d(j)x_j - x_jd(j) = \triangle(x_j),$$

for all j, where z_j is unit of $M_{I_{n_j}}$. Hence

$$\triangle(x) = \sum_j z_j \triangle(x) = \sum_j \triangle(x_j).$$

Since x was chosen arbitrarily $\triangle$ is a derivation on M by the last equality.

Indeed, let $x, y \in M$. Then

$$\triangle(x) + \triangle(y) = \sum_j \triangle(x_j) + \sum_j \triangle(y_j) = \sum_j [\triangle(x_j) + \triangle(y_j)] =$$

$$\sum_j \triangle(x_j + y_j) = \sum_j z_j \triangle(x+y) = \triangle(x+y).$$

Similarly,

$$\triangle(xy) = \sum_j \triangle(x_j y_j) = \sum_j [\triangle(x_j) y_j + x_j \triangle(y_j)] =$$

$$\sum_j \triangle(x_j) y_j + \sum_j x_j \triangle(y_j) = \sum_j \triangle(x_j) \sum_j y_j + \sum_j x_j \sum_j \triangle(y_j) =$$

$$\triangle(x) y + x \triangle(y).$$

By the proof of the previous theorem $\triangle$ is homogenous. Hence $\triangle$ is a linear operator and a derivation. The proof is complete. $\triangleright$

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